

ON CLASSES OF LOCAL UNITARY TRANSFORMATIONS

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ABSTRACT. We give a one-to-one correspondence between classes of density matrices under local unitary invariance and the double cosets of unitary groups. We show that the interrelationship among classes of local unitary equivalent multi-partite mixed states is independent from the actual values of the eigenvalues and only depends on the multiplicities of the eigenvalues. The interpretation in terms of homogeneous spaces of unitary groups is also discussed.

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1. INTRODUCTION

Theoretic foundation of quantum computation and quantum information [NC] has been postulated through the language of quantum mechanics. At the beginning of the development of quantum mechanics Schrödinger pointed out the unitary freedom of the quantum system [S, U], which is one of the spotlights in the whole theory and has been used in many applications in later developments. In [HJW] the freedom of purification of quantum states was studied with help of Schmidt decomposition. Furthermore the probability distributions given by a given density matrix are characterized by majorization [N]. These results have played an important role in quantum statistics, quantum computation and quantum information.

Quantum entanglement is one of the key issues in quantum computation, and local unitary invariance is an importance aspect in quantum entanglement and their applications. For instance, several fast algorithms are discovered based on special properties of local invariance and quantum entanglements. The maximum entanglement can also be achieved via local unitary action. Local unitary transformations

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have also been used for fractional entanglement and in other properties (cf. [HHH]). In various investigations of local unitary equivalence lots of efforts have been made in seeking possible invariants [AFG, TLT, ZSB, ACFW]. There have also been results on non-local unitary equivalence [DC] in exploring feasible quantum gates. In [YLF] a geometric class was proposed to study the problem for two partite quantum states and it has been shown that two quantum states are equivalent if and only if their representation classes are the same. In [FJ] we have proposed a new operational method to study local unitary equivalence of density matrices and the relationship with separability.

It is clear that in classification of equivalent density matrices one first needs to see whether two density operators have the same eigenvalues or not. Therefore one should separate the easy task of comparing eigenvalues from the problem and focus on other invariants of unitary operations. In the current paper we apply this strategy to a larger quantum system consisting of several quantum sub-systems and study their unitary properties from the viewpoints of both global and local pictures.

We first look at density matrices with specified spectrum of eigenvalues and study the properties of the quantum systems using group-theoretic methods. The local unitary properties are investigated with the help of fixed point subgroups of the concerned Lie group and we then set up a one-to-one correspondence between classes of local unitary equivalences and double cosets of the unitary group $U(n)$ by the additive and multiplicative Young subgroups. Next we recast the space of the equivalence classes in terms of induced representations of the Young subgroups and show their deep connections with homogeneous spaces of unitary groups.

One of our main results shows that local unitary equivalence depends only on the multiplicities of the eigenvalues and is independent from the actual values of the eigenvalues. This means that with given spectrum of the density matrices the local unitary equivalence is determined by the combinatorics of the quantum states. We hope that the combinatorics and invariant theory will shed light in further investigation of local unitary properties and entanglement of quantum states.

2. LOCAL UNITARY EQUIVALENCE

Let H be a Hilbert space affording the quantum system. If the quantum system is in a number of normalized quantum states $|\psi_i\rangle$ with probability p_i , then the density operator for the quantum system

is the convex combination

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|,$$

where $\sum_i p_i = 1$. Equivalently density operators are characterized as non-negative operators on the Hilbert space with unit trace.

Among all physically possible quantum systems afforded by H , one needs to judge whether two systems are distinct or not. To study the equivalence of various quantum systems on H , one only needs to consider the unitary linear group $U(n)$ over \mathbb{C} . Suppose two density matrices ρ_i have the same set of eigenvalues and their sets of multiplicities for each eigenvalue are also the same. Let the corresponding (orthonormal) eigenstates be $|\phi(\lambda_i^{(j)})\rangle$ and $|\psi(\lambda_i^{(j)})\rangle$ respectively, where $j = 1, \dots, m_i$, the multiplicity of the eigenvalue λ_i . Then we can write

$$\begin{aligned}\rho_1 &= \sum_{ij} \lambda_i |\phi(\lambda_i^{(j)})\rangle\langle\phi(\lambda_i^{(j)})|, \\ \rho_2 &= \sum_{ij} \lambda_i |\psi(\lambda_i^{(j)})\rangle\langle\psi(\lambda_i^{(j)})|.\end{aligned}$$

We define g to be the linear transformation on H sending $|\phi(\lambda_i^{(j)})\rangle \mapsto |\psi(\lambda_i^{(j)})\rangle$, then $g \in U(n)$ due to orthogonality of the eigenstates. Then one easily checks that

$$\rho_2 = g\rho_1 g^{-1} = g\rho_1 g^\dagger.$$

In other words density matrices with similar spectral decomposition are unitary equivalent and all physical features of the quantum system are captured by properties of the unitary group $U(n)$.

The above discussion can also be viewed in terms of group action. Let $D(n)$ be the set of density matrices on the Hilbert space H of dimension n . The evolution of the density operator ρ is given by $\rho \longrightarrow g\rho g^\dagger$, where $g = g(t) \in U(n)$ and \dagger is transpose and conjugation. Clearly $g\rho g^\dagger$ is positive and $\text{tr}(g\rho g^\dagger) = \text{tr}(\rho) = 1$. We will adopt the standard convention in group theory to write $hgh^{-1} = g^h$ for $g, h \in G$.

Lemma 2.1. *The kernel of the action $U(n) \times D(n) \longrightarrow D(n)$ is $\{e^{i\theta}I | \theta \in \mathbb{R}\} \leq U(n)$.*

Proof. Suppose $g\rho g^\dagger = \rho$ for all density matrices ρ . If ρ is any diagonal matrix with distinct eigenvalues, then g must be diagonal by computation. A generalized (or signed) permutation matrix is one that each row or column has only one and only non-zero entry and this non-zero entry is 1 or -1 . One notes that any matrix commutes with a generalized permutation matrix must be diagonal. To show that $g = aI$

with $a = e^{i\theta}$ a complex number of unit modulus, we can take positive generalized permutation matrix, then all diagonal entries of g must be equal. \square

For $\rho \in D(n)$ we define the *invariant subgroup* U_ρ by $U_\rho = \{g \in U(n) \mid g\rho g^\dagger = \rho\}$. As $g \in U(n)$ is unitary: $gg^\dagger = g^\dagger g = I$, we have immediately that $g\rho g^\dagger = \rho$ iff $g\rho = \rho g$. The next statement is immediate from definition.

Lemma 2.2. *For any $h \in GL(n)$ one has $U_{\rho^h} = hU_\rho h^\dagger = (U_\rho)^h$.*

Let ρ be a density matrix of size $n \times n$ and thus is diagonalizable due to hermiticity. According to basic theory in linear algebra [L] there exists a unitary matrix $g \in U(n)$ such that

$$\rho = g \operatorname{diag}(a_1 I_{\lambda_1}, \dots, a_l I_{\lambda_l}) g^\dagger,$$

where λ_i are non-negative integers such that $\lambda_1 + \dots + \lambda_l = n$. We can further suppose that λ_i are arranged in a descending order: $\lambda_1 \geq \dots \geq \lambda_l > 0$, thus λ is a partition of n . Here l is the number of parts of λ .

Definition 2.3. We say a density matrix ρ is of type λ , a partition of n , if ρ has eigenvalue multiplicities: $\lambda_1, \dots, \lambda_l$, where $\lambda_1 + \dots + \lambda_l = n$.

Two density matrices are *equivalent* if they are related by a unitary transformation: $\rho_1 = g\rho_2 g^\dagger$, $g \in U(n)$. This is in agreement with the equivalence relation given by the group action of $U(n)$ on $D(n)$. Thus the invariant subgroups of equivalent density matrices are conjugate in $U(n)$. We will call the set of equivalent density matrices of type λ the *equivalent class* of type λ , and denote the class by $[\lambda]$.

Proposition 2.1. *If $\rho = \operatorname{diag}(a_1 I_{\lambda_1}, \dots, a_l I_{\lambda_l})$ is a density matrix, then $U_\rho = U(\lambda_1) \times \dots \times U(\lambda_l)$. More generally if ρ is of type $\lambda = (\lambda_1, \dots, \lambda_l)$, then $U_\rho = gU(\lambda_1) \times \dots \times U(\lambda_l)g^\dagger$ for some $g \in U(n)$.*

Proof. If $\rho = \operatorname{diag}(a_1 I_{\lambda_1}, \dots, a_l I_{\lambda_l})$, it follows from direct computation that $U_\rho = U(\lambda_1) \times \dots \times U(\lambda_l)$. Then for a general density matrix ρ of type λ , we have that $\rho = g \operatorname{diag}(a_1 I_{\lambda_1}, \dots, a_l I_{\lambda_l}) g^\dagger$ ($g \in U = U(n)$), then the result follows from Lemma 2.2. \square

The classes of density matrices can be characterized by the group action.

Theorem 2.4. *Let $[\lambda]$ be the equivalent class of density matrices of type λ , then $[\lambda] \simeq U/U_\rho$, where ρ is some density matrix of type λ .*

Proof. Let $U = U(n)$. The action $U \times [\lambda] \longrightarrow [\lambda]$ given by $(g, \rho) \mapsto g\rho g^\dagger$ is transitive. As the class $[\lambda] = \{\rho^g | g \in U\}$ for a fixed density matrix ρ , we see that

$$\rho^{g_1} = \rho^{g_2} \iff g_1 g_2^{-1} \in U_\rho,$$

therefore $U/U_\rho \simeq [\lambda(\rho)]$. It follows from Proposition 2.1 that $U/U_\rho = U/gU_{\lambda(\rho)}g^{-1} \simeq U/U_{\lambda(\rho)}$ \square

We now consider local unitary equivalence. Let H_i be two Hilbert spaces with dimensions n_i . We say two density operators ρ_i on the space $H = H_1 \otimes H_2$ are equivalent under local transformation iff $\rho_1 = (g_1 \otimes g_2)\rho_2(g_1 \otimes g_2)^\dagger$ for some $g_i \in U(n_i) \leq \text{End}(H_i)$. We recall that ρ_i are (globally) equivalent if $\rho_1 = g\rho_2 g^\dagger$ for $g \in U(n) \leq \text{End}(H)$, where $n = n_1 n_2$. More generally, for multi-partite cases the local unitary group is $U_{\mathbf{n}} = U(n_1) \otimes \cdots \otimes U(n_r)$, where $n_i = \dim(H_i)$.

Theorem 2.5. *Let $H = H_1 \otimes \cdots \otimes H_r$ be the global Hilbert space with $\dim(H_i) = n_i$, and $U_{\mathbf{n}}$ as above. Then the type λ equivalent multipartite mixed states under local equivalence are in one-to-one correspondence to double cosets of the unitary group $U(n)$ by the Young subgroups: $U_{\mathbf{n}} \backslash U/U_\lambda$, where $U_\lambda = U(\lambda_1) \times \cdots \times U(\lambda_l)$. Moreover, the representative of the double coset determined by the density matrix ρ is given by the matrix of the orthonormal eigenstates of ρ .*

Proof. Globally equivalent density matrices are determined by their eigenvalue spectrum and multiplicities. The class of (locally) equivalent density matrices of type λ consists of

$$g\Lambda g^\dagger$$

where $\Lambda = \text{diag}(a_1 I_{\lambda_1}, \dots, a_l I_{\lambda_l})$ and g is a unitary matrix consisting of orthonormal eigenstates of ρ . Here Λ is fixed. When two density matrices $\rho_i = g_i \Lambda g_i^\dagger$ are local unitary equivalent, then $\rho_1 = k \rho_2 k^{-1}$ for some $k \in U_{\mathbf{n}}$. It then follows that

$$g_1 \Lambda g_1^\dagger = k g_2 \Lambda g_2^\dagger k^\dagger \implies g_1^{-1} k g_2 = c \in U_\Lambda,$$

and $U_\Lambda = U_\lambda = U(\lambda_1) \times \cdots \times U(\lambda_l)$ by Proposition 2.1. Therefore $U_{\mathbf{n}} g_1 U_\lambda = U_{\mathbf{n}} g_2 U_\lambda$.

Conversely, suppose two double cosets $U_{\mathbf{n}} g_i U_\lambda$ are the same. Then $g_1 = k g_2 c$ for some $k \in U_{\mathbf{n}}$ and $c \in U_\lambda$. It follows that $g_1^{-1} k g_2 = c^{-1} \in U_\lambda$, thus $g_1 \Lambda g_1^\dagger = k g_2 \Lambda g_2^\dagger k^\dagger$. It is clear that $\rho_1 = g_1 \Lambda g_1^\dagger$ gives rise to a density matrix and $\rho_2 = g_2 \Lambda g_2^\dagger$ gives rise to a globally equivalent density matrix (as $\rho_1 = \rho_2^k$). Their corresponding classes (under local equivalence) are also equal:

$$[g_1 \Lambda g_1^\dagger] = [g_2 \Lambda g_2^\dagger].$$

□

3. INDUCED REPRESENTATIONS

Let λ be a partition of n , $n = \lambda_1 + \cdots + \lambda_l$, and \mathbf{n} be a factorization of n , $n = n_1 \cdots n_r$, where n is the dimension of the underlying (global) Hilbert space. The Young subgroup U_λ associated with the partition λ is the direct product $U(\lambda_1) \times \cdots \times U(\lambda_l)$. From our previous discussion it is clear that one also needs to consider the multiplicative Young subgroup $U_{\mathbf{n}} = U(n_1) \otimes \cdots \otimes U(n_r)$ associated to the factorization \mathbf{n} of n . The additive Young subgroup $U_\lambda = U(\lambda_1) \times \cdots \times U(\lambda_l)$ can be imbedded into $U(n)$ in the canonical manner:

$$(g_1, \cdots, g_l) \in U(\lambda_i) \mapsto g_1 \times \cdots \times g_l = \begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_l \end{pmatrix} \in U(n),$$

while the multiplicative Young subgroup $U_{\mathbf{n}} = U(n_1) \otimes \cdots \otimes U(n_r)$ is imbedded into $U(n)$ via tensor product

$$g_1 \otimes \cdots \otimes g_r \in U_{\mathbf{n}} \mapsto g_1 \otimes \cdots \otimes g_r \in U(n).$$

As we have seen in Section 2 the double cosets given by the Young subgroups U_λ and multiplicative Young subgroup $U_{\mathbf{n}}$ are in one-to-one correspondence of quantum multi-partite systems with dimensions n_i . We now reformulate this correspondence in terms of representations of the unitary group. We can restrict ourself to the special unitary group without loss of generality. Let $G = SU(n)$ and $H = SU_\lambda$, and it is clear that H is a closed subgroup of the compact Lie group G .

Let \mathbb{C} be the trivial representation of the subgroup H . The induced representation $Ind_H^G \mathbb{C}$, as a vector space, is the space of all complex continuous functions on H satisfying the condition:

$$f(gh) = h^{-1}f(g), \quad g \in G, h \in H.$$

The action of G on the induced representation is given by

$$(3.1) \quad (g \cdot f)(x) = f(g^{-1}x), \quad g, x \in G.$$

The induced representation $Ind_H^G \mathbb{C}$ is isomorphic to $C^0(G/H, \mathbb{C})$, the space of continuous functions on the cosets G/H [BD]. Here the action is the left translation: $L(g)f(x) = f(g^{-1}x)$. The canonical basis of the representation is given by characteristic functions ϕ_{gH} , indexed by right cosets G/H . Here

$$(3.2) \quad \phi_{gH}(g'H) = \begin{cases} 1, & g^{-1}g' \in H \\ 0, & g^{-1}g' \notin H \end{cases}.$$

The induced representation $Ind_H^G \mathbb{C}$ is essentially equivalent to the set of unitary classes of density matrices as seen from the following observation.

Example 3.1. Suppose $n = 1 \cdot n$ is the factorization of n , then local unitary equivalence coincides with global unitary equivalence. The double cosets $U_{\mathbf{n}} \backslash U / U_{\lambda}$ reduces to cosets U / U_{λ} . The unitary equivalence classes (with same set of spectrum) are also in one-to-one correspondence of the set of partitions $\mathcal{P}(n)$:

$$n = \lambda_1 + \cdots + \lambda_l.$$

Each equivalence class is represented by a homogeneous space of the unitary group $U(n)$ determined by the partition λ . The generating function of the number of unitary equivalence classes is given by [M]

$$\sum_{n=0}^{\infty} \#(\text{equiv class}) q^n = \prod_{n=1}^{\infty} \frac{1}{1 - q^n}.$$

Example 3.2. Density matrices on any Hilbert space of prime dimension are always inseparable. Multipartite states should live on Hilbert spaces of dimension equal to powers of prime numbers.

Local unitary equivalent classes can also be studied from representation theoretic viewpoints. Let $K = SU_{\mathbf{n}}$ corresponding to the factorization \mathbf{n} , and $H = SU_{\lambda} \leq G = SU(n)$ as above. Both H and K are closed subgroups of G . We consider the restriction $Res_K Ind_H^G \mathbb{C}$. Let W be any H -module, then it follows from Frobenius reciprocity [BD] that

$$\begin{aligned} Hom_K(Res_K Ind_H^G \mathbb{C}, W) &\simeq Hom_G(Ind_H^G \mathbb{C}, Ind_K^G W) \\ &\simeq Hom_H(\mathbb{C}, Res_H Ind_K^G W). \end{aligned}$$

In particular, when $W = \mathbb{C}$, one has the duality $Res_K Ind_H^G \mathbb{C} \simeq Res_H Ind_K^G \mathbb{C}$. The linear operators in $Hom_G(Ind_H^G \mathbb{C}, Ind_K^G \mathbb{C})$ in general are certain distributions on the direct product of the unitary group by Schwartz's distribution theory [Sc]. If the intertwining number (the dimension of $Hom_G(Ind_H^G \mathbb{C}, Ind_K^G W)$) is finite, then they are equal to the number of double cosets of $K \backslash G / H$ by Mackey's theorem [Mc]. Thus we obtain the following result.

Theorem 3.3. *Let $H = H_1 \otimes \cdots \otimes H_r$ be the global Hilbert space with $dim(H_i) = n_i$, and $U_{\mathbf{n}}$ as above. Then the classes of globally equivalent multi-partite mixed states under local equivalence are isomorphic to the restriction of the induced representation $Ind_{U_{\lambda}}^{U(n)} \mathbb{C}$ to the subgroup $SU_{\mathbf{n}}$.*

4. HOMOGENEOUS SPACES

Homogeneous spaces are important non-empty manifolds with a transitive action of a Lie group. The quotient space G/H are special examples of homogeneous spaces. If one views equivalent local unitary classes as points in the homogeneous spaces one may understand the geometric meaning of the statement that all points are the same.

Let \mathbf{n} be a fixed prime decomposition of n and λ be a fixed partition of n . If two density matrices ρ_i are locally equivalent, then their corresponding double cosets $U_{\mathbf{n}}g_1U_{\lambda}$ and $U_{\mathbf{n}}g_2U_{\lambda}$ are the same. Note that in this description the equivalence relations are determined completely by the double cosets, and no information about the actual eigenvalues are needed. In other words, classes of density matrices with the same eigenvalue distribution can be mapped to classes of density matrices with the same eigenvalue multiplicities. Therefore we have proved the following result.

Theorem 4.1. *Let $H = H_1 \otimes \cdots \otimes H_r$ be the global Hilbert space with $\dim(H_i) = n_i$, and $U_{\mathbf{n}}$ as above. Then the class of globally equivalent multi-partite mixed states under local equivalence can be mapped to another class of globally equivalent multi-partite mixed states with the same eigenvalue distribution and multiplicities. The two classes are both described by the homogeneous spaces $U_{\mathbf{n}} \backslash U/U_{\lambda}$, where the partition λ is given by the eigenvalue multiplicities.*

This result shows that the local unitary equivalence relations does not depend on the actual values of the eigenvalues, and all classes can be viewed as “equal” in geometric sense. Therefore we can parameterize the classes of density matrices by the set

$$\mathbb{R}^n \times (U_{\mathbf{n}} \backslash U(n)/U_{\lambda}),$$

where \mathbb{R} is used for the eigenvalue.

Example 4.2. For $e \geq 0$ and $0 \leq f \leq 1 - e$ we consider the two qubit Werner state

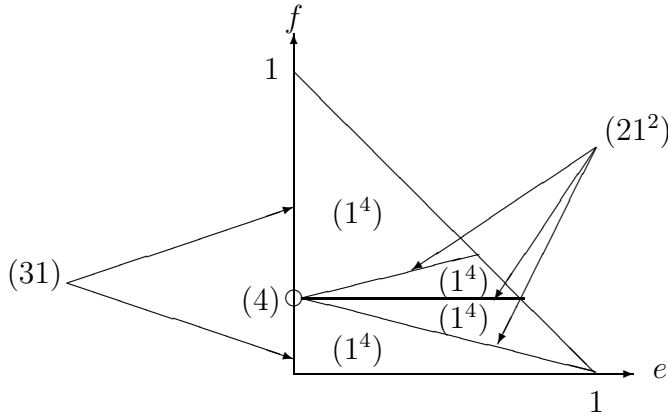
$$\rho = \begin{pmatrix} \frac{1-e-f}{3} & & & \\ & \frac{1+2f}{6} & \frac{1-4f}{6} & \\ & \frac{1-4f}{6} & \frac{1+2f}{6} & \\ & & & \frac{1+e-f}{3} \end{pmatrix}.$$

When $e = 0$, this is the usual Werner state [W]. The eigenvalues are

$$(1 - f + e)/3, (1 - f)/3, (1 - f - e)/3, f.$$

The nontrivial factorization is certainly $4 = 2 \cdot 2$ and the partition of the eigenvalue multiplicities are $\lambda = (1111)$ for $e > 0$ and $f \neq \frac{1+e}{4}$.

When $e > 0$ and $f = \frac{1+e}{4}$ or $1/4$ then $\lambda = (211)$. If $e = 0$, then $\lambda = (31)$ for $f \neq 1/4$ and $\lambda = (4)$ for $f = 1/4$. From the picture there are essentially one dense class of local unitary equivalence; two one-dimensional classes of unitary equivalence; and one degenerate class of unitary equivalence. The most interesting cases are $\lambda = (31)$ and (21^2) , where the local unitary classes are classified by double cosets of $SU_{(22)} \backslash SU(4) / (SU(3) \times SU(1))$ and $SU_{(22)} \backslash SU(4) / (SU(2) \times SU(1) \times SU(1))$ respectively. All classes with the same partition are viewed as equal.



5. CONCLUSION

We have established a one-to-one correspondence between the set of density matrices of multi-partite systems and the set of double cosets of additive and multiplicative Young subgroups of unitary groups. Our results show that the interrelationship among the classes of local unitary equivalent multi-partite mixed states is independent from the actual values of the eigenvalues and only depends on the multiplicities of the eigenvalues. This interesting phenomenon allows us to look at the local unitary equivalence for all global unitary density matrices at the same time. The geometry of the classes are given by that of homogeneous spaces of the unitary group as well as the invariant theory of classical groups (cf. [H]). It is expected that the combinatorics of density matrices and invariant theory will also play a role in this study.

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